Assignment 2

Hand in no. 1, 6 and 7 by September 26 .

1. A bounded function f on $[a, b]$ is said to be locally Lipschitz continuous at $x \in [a, b]$ if there exist some L and δ such that

$$|f(y) - f(x)| \le L|x - y| \ , \quad \forall y \in (x - \delta, x + \delta).$$

   Show that f is Lipschitz continuous at x.

2. Let f be a function defined on (a, b) and $x_0 \in (a, b)$.
   (a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.
   (b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

3. Provide a proof of Theorem 1.6.

4. (a) Show that the Fourier series of the function $\cos tx$, $x \in [-\pi, \pi]$ where t is not an integer is given by

$$\frac{\pi \cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx, \quad x \in [-\pi.\pi].$$

   (b) Deduce that for $t \in (0, 1)$,

$$\log \sin t\pi = \log t\pi + \sum_{n=1}^{\infty} \log\left(1 - \frac{t^2}{n^2}\right).$$

   (c) Conclude that

$$\frac{\sin t\pi}{\pi t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2}\right), \quad t \in (0, 1).$$

5. Can you find a cosine series which converges uniformly to the sine function on $[0, \pi]$? If yes, find one. You may use Theorem 1.7.

6. A sequence $\{a_n\}, n \ge 0$, is said to converge to a in mean if

$$\frac{a_0 + a_1 + \cdots + a_n}{n + 1} \to a \ , \quad n \to \infty \ .$$

   (a) Show that $\{a_n\}$ converges to a in mean if $\{a_n\}$ converges to a.
   (b) Give a divergent sequence which converges in mean.

7. Let D_n be the Dirichlet kernel and define the Fejer kernel to be $F_n(x) = \dfrac{1}{n+1} \sum_{k=0}^{n} D_k(x)$.
   (a) Show that

$$F_n(x) = \frac{1}{2\pi(n+1)} \left(\frac{\sin(\frac{n+1}{2})x}{\sin x/2}\right)^2 , \quad x \ne 0 \ .$$

(b) Let

$$\sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) \ .$$

Show that for every $x \in [-\pi, \pi]$, $\sigma_n f(x)$ converges uniformly to $f(x)$ for any continuous, 2π-periodic function f. Hint: Follow the proof of Theorem 1.5 and use the non-negativity of F_n.